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Gravitational Collapse and Ergodicity in Confined Gravitational Systems

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The ergodic properties of many-body systems with repulsive-core interactions are the basis of classical statistical mechanics and are well established. This is not the case for systems of purely-attractive or gravitational particles. Here we consider two examples, (i) a family of one-dimensional systems with attractive power-law interactions, $|x_i - x_j|^\nu$, $\nu > 0$, and (ii) a system of N gravitating particles confined to a finite compact domain. For (i) we deduce from the numerically-computed Lyapunov spectra that chaos, measured by the maximum Lyapunov exponent or by the Kolmogorov–Sinai entropy, increases linearly for positive and negative deviations of ν from the case of a non-chaotic harmonic chain ($\nu = 2$). For $2 < \nu \leq 3$ there is numerical evidence for two additional hitherto unknown phase-space constraints. For the theoretical interpretation of model (ii) we *assume* ergodicity and show that for a small-enough system the reduction of the allowed phase space due to any other conserved quantity, in addition to the total energy, renders the system asymptotically stable. Without this additional dynamical constraint the particle collapse would continue forever. These predictions are supported by computer simulations.

KEY WORDS: Unstable systems, confined gravitational systems, ergodicity, chaotic dynamics, conservation laws, lyapunov spectrum

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1. INTRODUCTION AND MOTIVATION

A very basic assumption of classical equilibrium statistical mechanics is that the phase trajectory originating from an (almost) arbitrary point in phase space

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densely covers a representative part of the whole energy surface such that any time average of an observable can be replaced by an ensemble average over the energy surface. From a physical point of view, this so-called *ergodic hypothesis*[1] turns out to be extremely successful for three- or two-dimensional molecular systems with a short-ranged repulsive pair potential. Even for three particles in a box or on a torus numerical simulations suggest ergodicity and mixing. Mathematically, ergodicity and strong mixing has only been proven for a class of hard ball systems,[2] such as the fully hyperbolic Sinai billiards with strictly convex scatterers,[3] and for a large number of hard spheres on a three-dimensional torus.[4] Unfortunately, these proofs do not necessarily carry over to more physical systems with smooth interactions,[5] but they carry over to infinite quantum systems.[6]

The situation is even more complicated for systems with long range interactions such as gravitation, and not much is known for such cases.[7,8] This is partly due to the fact that in astrophysical problems the phase space is usually not bounded and that the notion of ergodicity is more problematic than in statistical mechanics.

Here, we examine two problems, which nicely illustrate the complexity of the situation.

The first of these problems is the one-dimensional sheet model,[9] which was originally proposed as a model for the dynamics of stars transverse to the galactic plane of a highly-flattened galaxy.[10,11] An exact statistical description has been given by Rybicki, both in the canonical and microcanonical ensembles.[12] Here we are particularly interested in the microcanonical case, which parallels our computer simulations of the Lyapunov spectrum below. The model consists of N infinite parallel mass sheets, where each sheet extends over the whole yz-plane and moves along the x-axis under the mutual gravitational attraction of all the other sheets. The Hamiltonian is usually written in the form

$$\mathcal{H} = \sum_{i=1}^{N} \frac{\tilde{p}_i^2}{2s} + 2\pi G s^2 \sum_{i=1}^{N-1} \sum_{j>i}^{N} \left| x_i - x_j \right| \ , \tag{1}$$

where G is the gravitational constant, x_i and $\tilde{p}_i$ denote the position and momentum of a sheet, and s is its mass. Due to the attractive potential the final state is a single cluster, and no external boundary is required. Assuming microcanonical equilibrium, the local density and the velocity distributions are known[9] for any number of particles, N. However, for $N \leq 10$ Reidl and Miller[13,14] have demonstrated the existence of regularity islands with a finite phase space measure, which strictly prevents ergodicity in this case. For the computation of single-body equilibrium properties[15] the statistical significance of these islands is small: for fully-relaxed equilibrium systems with $N = 10$ particles the density and velocity distributions of the cluster particles agree well with the respective theoretical

distributions based on the assumption of ergodicity.[9] For the time evolution, however, the islands play a decisive role by trapping evolving states for a long time. Since they are surrounded by "sticky regions" with almost vanishing hyperbolicity and, hence, spreading power between (infinitesimally) separated trajectories, the computation of the time-dependent (local) Lyapunov exponents is a convenient tool for the localization of such sticky regions and hidden regularity islands. From our simulations we have indications for the existence of such sticky regions for systems with up to 32 sheets.[9]

In Sec. 2 we extend this work to a whole family of attractive potentials which contains the sheet model as a special case. The emphasis is on the Lyapunov spectrum and the Kolmogorov–Sinai entropy of such systems, which are taken as a measure of chaos. In addition to the familiar constants of the motion, we present evidence for strange dynamically-conserved quantities in phase space which cause two additional Lyapunov exponents to vanish.

Recently we have studied the famous restricted three-body problem with a heavy mass M (the "Sun") and a small mass m ("Jupiter") orbiting with constant angular velocity in a plane around their common center of mass. The motion of a test particle ("Satellite") in the orbital plane of M and m is observed, which is so light that it does not affect their circular orbits. Let us consider a strongly perturbed case $m/M = 1/9$[16] (which exceeds the actual mass ratio between Jupiter and the Sun by more than a factor of 100). By a suitable choice of the test particle's energy it moves chaotically in a co-rotating bounded domain in space which includes the regions around M and m with a narrow connecting channel between them. If the system is ergodic, a simple theorem[16] requires that the microcanonical density in configuration space is constant over the whole accessible domain. Computer simulations reveal that the density is almost, but not strictly, constant indicating that the system is not strictly ergodic. This is confirmed by the existence of regularity islands in a Poincaré map of the flow.[16]

This problem has motivated us to consider as our second example in Sec. 3 the general (unrestricted) case of $N \geq 3$ particles of similar or equal mass in the plane, which interact with a gravitational $1/r$-potential and which are confined to a bounded domain. We distinguish three cases:

1. The confining boundary is a reflecting square such that the energy is the only conserved quantity.
2. The confining boundary is a reflecting circle such that energy and angular momentum are conserved.
3. The particles are on a torus T^2 (periodic boundaries) such that energy and linear momentum are conserved.

Assuming that the systems are ergodic and that the microcanonical equilibrium state exists, it will be shown in Sec. 3.1 that, for small-enough N, the phase-space

reduction associated with any conserved quantity, in addition to the total energy, is responsible for a qualitatively-different behavior between case 1 and the other cases 2 and 3. These microcanonical predictions are tested and confirmed, to some extent, by computer simulations in Sec. 3.2. We conclude with a short discussion in Sec. 4.

2. CHAOS IN ONE-DIMENSIONAL SELF-GRAVITATING SYSTEMS

In this section we consider N particles in one dimension with a Hamiltonian

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \frac{\lambda}{N} \sum_{i=1}^{N-1} \sum_{j>i}^{N} \left| \frac{x_i - x_j}{\sigma} \right|^{\nu}, \tag{2}$$

where x_i, p_i and m in Eq. (2) are the position, momentum and mass of particle i, respectively, and where ν is a positive parameter.[9] For $\nu = 1$, and in view of the scaling relations $\tilde{p} = p/N$, $s = m/N$, and $\mathcal{H} = H/N$, Eq. (2) reduces to the sheet-model Hamiltonian in Eq. (1), if λ is identified with $2\pi G s^2$. For $\nu = 2$ the Hamiltonian (2) describes a linear harmonic chain, for $\nu = 0$ it corresponds to a one-dimensional ideal gas, and for $\nu = -2$ to an exactly-solvable Calogero–Sutherland model on the line,[17,18] which has found many recent applications in physics.[19] We restrict our simulations to $\nu \geq 1$.

In our numerical work we use reduced units for which the length parameter σ, the particle mass m, and the energy parameter λ are all unity. The unit of time is $\sqrt{m\sigma^2/\lambda}$. In all numerical simulations the energy per particle, H/N, is also unity, and the total energy becomes $E = N\lambda$.

2.1. The Sheet Model, $\nu = 1$

For all simulations with $\nu = 1$ we use an "exact" algorithm, which steps from one crossing event of any two particles to the next,[20,21] and which uses a highly-optimized sorting scheme of order $\mathcal{O}(N)$.[22] More recently, heap-ordered tree codes have been suggested.[23] Here, we compute full Lyapunov spectra for systems with $N \leq 90$. Each system is relaxed for at least 7×10^7 time units before an averaging for the Lyapunov spectrum is initiated, which lasts for at least 1.8×10^6 time units. For comparison, an oscillation period t_c for a typical particle is about 7.3 time units. Our algorithm for the computation of the Lyapunov spectrum is described in detail in ref. 9.

In Fig. 1 the particle-number dependence of the maximum Lyapunov exponent, λ_1, and that of the Kolmogorov–Sinai entropy per particle, h_{KS}/N, is shown. According to Pesin's theorem[24] h_{KS} is obtained as the sum of all positive exponents. Figure 1 extends our previous results[9] to $N = 90$. Where the data overlap, the agreement with the results of Tsuchiya *et al.*[25,26] and Benettin

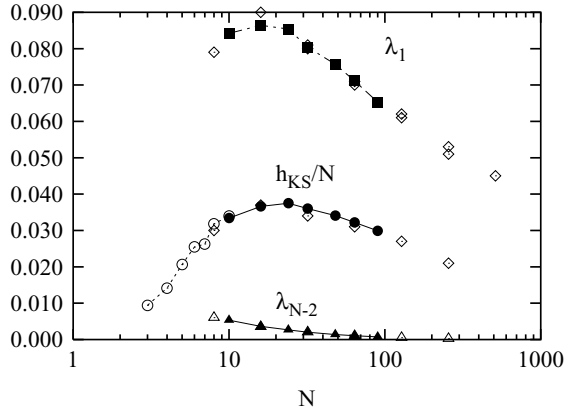

Fig. 1. Particle-number dependence of the maximum Lyapunov exponent, λ_1, of the Kolmogorov–Sinai entropy per particle, $h_{\rm KS}/N$, and of the smallest positive exponent, λ_{N-2}, for the sheet model ($v = 1$). The open squares, diamonds, and triangles are results taken from Tsuchiya *et al.* and the open circles from Benettin *et al.* as explained in the main text.

et al.[29](distinguished by the open symbols) is excellent. One finds that the strongest phase-space expansion occurs for $N \approx 16$. The Kolmogorov–Sinai entropy per particle has a maximum for $N \approx 24$. There is supportive evidence by Reidl and Miller, who found that clusters of about that size coalesce most rapidly[31] indicating fastest mixing.

As expected, both λ_1 and $h_{\rm KS}$ decrease monotonously for larger N and vanish for $N \to \infty$ due to the regularity of the mean-field solution in this limit. For systems in maximum-entropy equilibrium states, the data of Tsuchiya and Gouda[26] (open diamonds and triangles in Fig. 1) suggest the following asymptotic scaling behavior for large N: $\lambda_1 \sim 1/N^{1/5}$, $\lambda_{N-2} \sim 1/N$, and $h_{\rm KS}/N \sim 1/N^{1/5}$. However, in all these simulations no full convergence of the shape of the Lyapunov spectrum to a universal scaling function has been achieved, as N becomes large. Since $h_{\rm KS}$ converges to zero for $N \to \infty$, phase-space mixing becomes progressively insignificant and the relaxation of collective properties[15] to equilibrium becomes excessively slow.[30]

As noted also by Tsuchiya *et al.*[26] the perturbation vector associated with λ_1 is strongly localized in physical space and in the single-particle phase space, such that only a small fraction of all the particles contributes to the respective instability at any instant of time. The same result also applies for systems with repulsive interactions.[27,28] However, the smallest positive exponents (such as λ_{N-2}) have perturbation vectors with non-vanishing components contributed by *all* the particles and, thus, describe *collective* instabilities in phase space.[26] In a system of hard particles confined to a periodic or reflecting square box, such perturba-

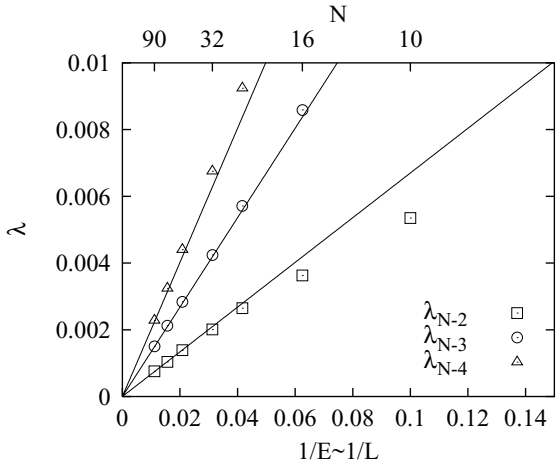

Fig. 2. Energy dependence for the three smallest positive Lyapunov exponents, λ_{N-4}, λ_{N-3} and λ_{N-2}, of the sheet model ($\nu = 1$). The respective limiting slopes are 1×0.067, 2×0.067, and 3×0.067.

tion vectors even generate Lyapunov modes, which are periodic perturbations in physical space with a well-defined wave vector and reminiscent of the Goldstone modes of fluctuating hydrodynamics.[28,32–35] Although we did *not* find Lyapunov modes for the sheet model, we made the following observation. Let $L/2$ denote the maximum possible excursion of a particle, when all the other particles are at rest at a single point, $L/2 = E\sigma/\lambda \sim N$. A plot of the smallest positive exponents, λ_{N-4}, λ_{N-3} and λ_{N-2}, in Fig. 2 as a function of $1/E$ ($\sim 2/L$) reveals straight lines for large systems, $E \to \infty$, which *resemble* the linear "dispersion relations" for the Lyapunov modes found in quasi-one-dimensional hard-disk fluids.

2.2. The Case $\nu > 1$

For $\nu \neq 1$ we have to solve the equations of motion explicitly. A predictor-corrector algorithm,[36] accurate to $O((\Delta t)^5)$, with a fixed time step $\Delta t = 2^{-10}$, is used. Such a small time step is necessary for an accurate localization of the crossing events. The time evolution of the Lyapunov vectors is computed with a Runge–Kutta algorithm and with the same time step. Excluding the harmonic case $\nu = 2$, the relaxation to microcanonical equilibrium is much faster than for the systems with $\nu = 1$.[9] Thus, the initial relaxation period may be reduced to about 8000 time units. It is followed by an averaging interval for the Lyapunov spectrum of about 2×10^5 time units.

In Fig. 3 the maximum Lyapunov exponent, λ_1, and the Kolmogorov–Sinai entropy per particle, h_{KS}/N, of a 16-particle system are shown as a function of

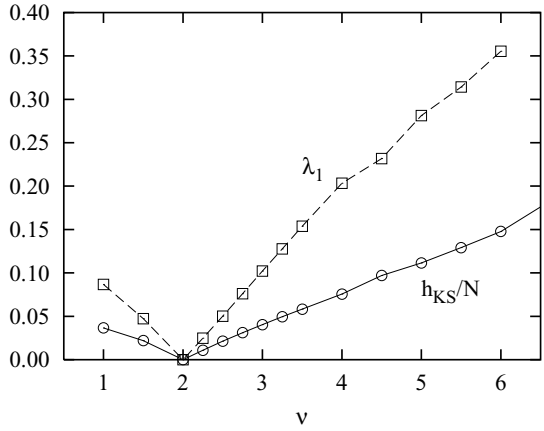

Fig. 3. Plot of the maximum exponent, λ_1, and of the Kolmogorov–Sinai entropy per particle, h_{KS}/N, for a 16-particle system as a function of the power ν of the pair potential.

the potential parameter ν. As expected, both quantities vanish for $\nu = 2$. This is the case of a regular harmonic chain, which is never chaotic. However, it is very interesting to note that chaos, measured by λ_1 or h_{KS}, increases *linearly* for both positive and negative deviations of ν from this specific case. The slope for h_{KS}/N is 0.037 for $\nu > 2$.

It is interesting to note that for $\nu > 2$ the maximum exponent λ_1 is insensitive to N: the results for a 10-particle system (not shown in Fig. 3) agrees very well with the 16-particle result displayed in Fig. 3. This is a similar result we found for hard-disk systems in two dimensions known to be ergodic.[32,34] It means that the maximum Lyapunov exponent, which describes the fastest phase-space expansion, is basically determined by two-body interactions and is not significantly affected by the collective dynamics for $\nu > 2$. For $\nu < 2$, and for the sheet model, $\nu = 1$, in particular, λ_1 strongly depends on N, which points to the collective nature of the dynamics, culminating in the mean-field result $\lim_{N\to\infty} \lambda_1 = 0$.

In Fig. 4 we plot the two *smallest positive* exponents, λ_{N-3} and λ_{N-2}, for two systems containing $N = 16$ (full symbols) and $N = 10$ (open symbols) particles, as a function of ν. λ_{N-3} (full and open squares) behaves as expected, being positive for all ν with the exception of the case of the harmonic chain, $\nu = 2$. But it is most surprising to note that for both systems the exponent λ_{N-2} (full and open circles), which is expected to be small but positive, also *vanishes*, within numerical accuracy, for $2 < \nu \leq 3$. The same is true for the conjugate exponent, λ_{N+3}, which according to the conjugate pairing rule for symplectic systems [37–39] is equal to $-\lambda_{N-2}$. For $2 < \nu \leq 3$, the systems behave as if, in addition to translation invariance in space and time-shift invariance, there existed another independent

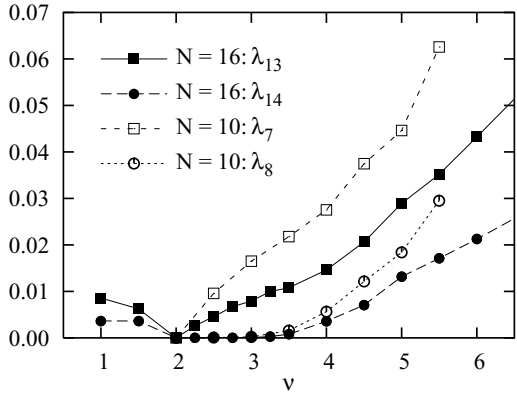

Fig. 4. ν-Dependence of the two smallest positive Lyapunov exponents for a 16-particle system, λ_{13} (full squares) and λ_{14} (full circles), and for a 10-particle system, λ_7 (open squares) and λ_8 (open circles).

continuous symmetry, which, according to Nöther's theorem,[40,41] could give rise to the two additional phase-space constraints and, hence, to the two vanishing exponents unaccounted for up to now. Translation invariance in space, responsible for momentum and center-of-mass conservation, generates two of the regularly vanishing exponents, time-shift invariance, responsible for energy conservation and vanishing perturbation growth in flow direction, generates the other two.[41] In spite of a considerable effort, we have not been able to account for the surplus vanishing exponents for $2 < \nu \leq 3$. This result seems to be independent of N.

3. GRAVITATIONAL PARTICLES CONFINED TO A BOX

3.1. Microcanonical Theory

We consider the case of N point particles with masses m_i in d dimensions, and introduce a one-parameter family of Hamiltonians

$$H = \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m_i} + V; \quad V = -G_\nu \sum_{i<j} \frac{m_i m_j}{|\mathbf{x}_i - \mathbf{x}_j|^\nu}, \tag{3}$$

with a positive parameter ν. Here, $\mathbf{x}_i$ and $\mathbf{p}_i$ denote the position and momentum of particle i, and G_ν is the associated gravitational constant. In what follows we *assume* that the system is ergodic. We distinguish three cases:

(1) The energy is the only conserved quantity, $H = E$, and linear and angular momenta are not treated explicitly. The probability measure in phase space is then given by $\delta(H - E)e^{-S(E)}$, where $S(E)$ is the microcanonical

entropy. The state integral becomes

$$\int d^d p_1 \cdots d^d p_N \int_\Omega d^d x_1 \cdots d^d x_N \; \delta\left(\sum_i \frac{\mathbf{p}_i^2}{2m_i} + V - E\right)$$

$$= c_1 \int_\Omega d^d x_1 \cdots d^d x_N \left| E + G_\nu \sum_{i<j} \frac{m_i m_j}{|\mathbf{x}_i - \mathbf{x}_j|^\nu} \right|_+^{(dN/2)-1} . \qquad (4)$$

where c_1 is a constant. The space integral, which is over a finite compact domain Ω, diverges for $\nu \geq \nu_c$, and remains finite for $\nu < \nu_c$, where the critical value ν_c is given by

$$\nu_c = \frac{2d}{dN - 2}. \qquad (5)$$

Thus, in two dimensions and for $\nu = 1$, the integral diverges for three or more particles.

(2) Energy and angular momentum are conserved. The angular momentum vector $\mathbf{L}$ is a pseudo vector with $d(d-1)/2$ components. If in the state integral the momenta and positions are constrained according to $\sum_i \mathbf{x}_i \wedge \mathbf{p}_i = \mathbf{L}$, where it is assumed that the total angular momentum vanishes, $\mathbf{L} = \mathbf{0}$, one finds

$$\int d^d p_1 \cdots d^d p_N \int_\Omega d^d x_1 \cdots d^d x_N \; \delta\left(\sum_i \frac{\mathbf{p}_i^2}{2m_i} + V - E\right) \delta^{d(d-1)/2}$$

$$\times \left(\sum_i \mathbf{x}_i \wedge \mathbf{p}_i\right) = \int_\Omega d^d x_1 \cdots d^d x_N \; c_2(x)$$

$$\times \left| E + G_\nu \sum_{i<j} \frac{m_i m_j}{|\mathbf{x}_i - \mathbf{x}_j|^\nu} \right|^{\frac{d}{2}\left[N - \frac{d-1}{2}\right]-1} . \qquad (6)$$

Here, $c_2(x)$ is a function of the coordinates, and Ω is again a finite compact domain in space. The convergence, or divergence, of the last integral is controlled by a critical value of ν given by

$$\nu_c = \frac{2d}{d\left[N - \frac{(d-1)}{2}\right] - 2}. \qquad (7)$$

For $d = 2$ and $\nu = 1$, the state integral is finite for three particles, but diverges for $N \geq 4$.

(3) Energy and linear momentum are conserved. In the state integral the momenta are constrained according to $\sum_i \mathbf{p}_i = \mathbf{P}$, where we assume that the total momentum $\mathbf{P}$ vanishes. One has

$$\int d^d p_1 \cdots d^d p_N \int_\Omega d^d x_1 \cdots d^d x_N \; \delta\left(\sum_i \frac{\mathbf{p}_i^2}{2m_i} + V - E\right) \delta^d\left(\sum_i \mathbf{p}_i\right)$$
$$= c_3 \int_\Omega d^d x_1 \cdots d^d x_N \left| E + G_\nu \sum_{i<j} \frac{m_i m_j}{|\mathbf{x}_i - \mathbf{x}_j|^\nu} \right|^{[d(N-1)/2]-1}, \tag{8}$$

where c_3 is a constant. Now the critical value for ν is

$$\nu_c = \frac{2d}{d(N-1)-2}. \tag{9}$$

For $d = 2$ and $\nu = 1$, the state integral diverges for $N \geq 4$.

Thus, assuming ergodicity we find for $\nu = 1$ that there is a qualitative difference for a planar three-body systems, whether there is a constant of the motion in addition to the total energy, or not. In the first case the state integral is finite and, as a consequence, any collapse of the three particles cannot be complete. The system heats up because of the collapse, but it reaches a stationary state with a finite kinetic energy. In the second case, however, no finite state can exist, and the system continues to collapse indefinitely.

In the following we test some of these theoretical predictions by computer simulations.

3.2. Computer Simulations

For our numerical work the particle masses are chosen to be similar, but not identical. We use reduced units, for which one of these masses is taken to be unity. The unit of length is given by the size of the box, and the unit of energy is then determined by taking the gravitational constant G_ν equal to unity. We restrict ourselves to the strictly gravitational case $\nu = 1$.

The planar systems considered here consist of three particles enclosed in a finite box with either hard or soft reflecting boundaries. For square boxes the energy is the only constant of the motion, which corresponds to the case (1) in Sec. 3.1. For circular boxes also the total angular momentum with respect to the box center is conserved, which conforms to our case (2). In principle, the conservation of linear momentum, case (3) of Sec. 3.1, can be achieved with periodic boundaries, but in practice such boundaries are much harder to implement for long-range gravitational potentials. They require the use of Ewald sums,[42,43]

which are not well suited for the small systems considered here. For that reason we have restricted ourselves to square and circular boundaries, the cases (1) and (2), respectively.

The gravitational potential (3) provides another challenge for the simulation, namely the singularity for vanishing particle separations. Various regularization schemes have been devised, of which the transformation to Kustaanheimo–Stiefel coordinates is the most successful.[44–46] Our simulation with soft boundaries uses this scheme and is based on a code originally supplied by Mikkola and Aarseth.[46] A Bulirsch–Stoer integration scheme[47] with variable time steps allows us to keep the energy error low even for trying triple collisions. Typically this error did not exceed 10^{-6} for a time of 150. The soft external box potentials are chosen according to

$$V(x, y) = x^{10} + y^{10} \tag{10}$$

for the square case, and

$$V(x, y) = \left[\frac{\pi}{4}(x^2 + y^2)\right]^5 \tag{11}$$

for the circular case. In Fig. 5 the time evolution of the kinetic energy, $\langle K(t)\rangle$, averaged over an ensemble of random initial conditions is shown for the two cases, where the total energy is -1. $\langle K\rangle$ continues to rise for the external potential with four-fold symmetry, whereas it saturates for the circular case. In the former case

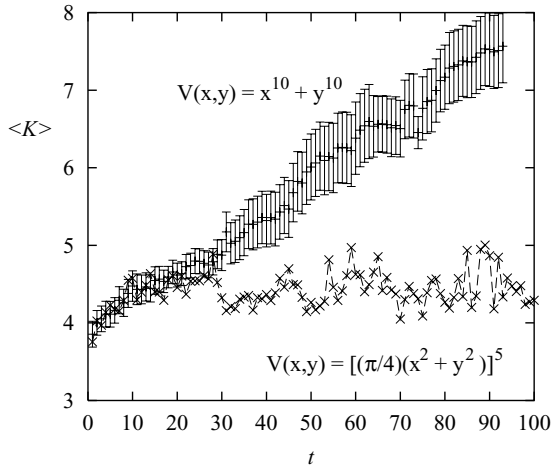

Fig. 5. Kinetic energy of a three-particle system in an external potential as indicated by the labels. The data are averages over an ensemble of 1000 trajectories with random initial conditions as described in the main text.

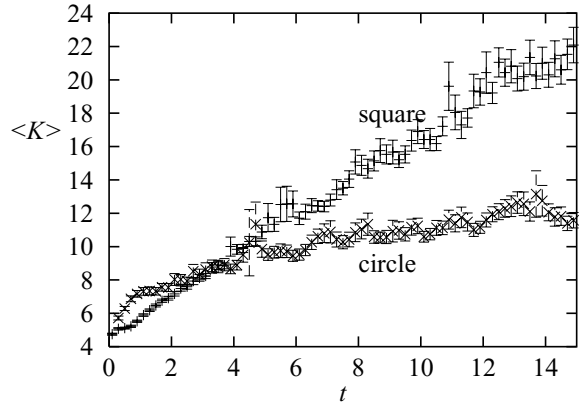

Fig. 6. Kinetic energy of a three-particle system in a box with an elastic hard boundary as indicated by the labels. The data are averages over an ensemble of 1000 trajectories with random initial conditions as described in the main text.

the collapse of the particles continues, whereas in the latter it comes to an end fairly soon.

For the simulations employing hard elastic boundaries another algorithm was used, which is based on a simpler, but less-efficient regularization scheme of Szebehely.[48] The equations of motion were integrated with a Runge–Kutta fourth-order algorithm with variable time step, and the energy drift did not exceed 10^{-4} over the length of a trajectory. The results are shown in Fig. 6.

Both regularization schemes used here fail for many-body collisions. Once they occur, phase-space trajectories have to be discontinued. As a consequence, the diverging kinetic energies could not be followed to times much larger than shown in Figs. 5 and 6.

The quantity actually computed and integrated together with the trajectory is $\langle \int_{\tau=0}^{t} K(\tau) d\tau \rangle_0$, where $K(t)$ is either the instantaneous kinetic energy with respect to the center of mass in the case of the smooth external potential, or the instantaneous total kinetic energy for the case of hard boxes. The bracket in this expression is an average over an ensemble of initial conditions, chosen from a uniform random distribution of the particle coordinates in the allowed spatial domain, and from a uniform random distribution of the momenta subject to a possible additional constraint (vanishing angular momentum with respect to the center of the box).The (numerical) derivative of this expression is taken as an estimate of the ensemble-averaged kinetic energy, $\langle K(t) \rangle$ at time t. Each ensemble average involves an average of typically thousand trajectories. The error bars in Figs. 5 and 6 correspond to estimates of the standard deviaton.

4. DISCUSSION

If the state integral of Sec. 3.1 diverges, we may be assured that the cluster of particles will eventually collapse. According to a classical theorem (Corollary (4.5.11) of ref. 50) we can say more: If the collapse occurs, it cannot be asymptotic, but must occur within a finite time. However, the theorem does not tell us how fast. For systems with many particles it was found that for energies above a certain threshold the system may be trapped in metastable states for a long time before it continues to collapse[49] and the gravothermal catastrophe occurs. Random fluctuations in combination with angular momentum exchange between the particles may even drive the system out of the metastable state into long-lasting gravo-thermal oscillations, before the collapse continues.

In a recent paper Pietronero and co-workers[51] studied the gravitational clustering of up to 32,000 mass points in three dimensions in a finite box with periodic boundaries. They find that the system finally settles in a stationary state characterized by a single cluster with a finite density distribution and floating in a gas of non-condensed particles. For so many particles ergodicity may be taken for granted, and our case (3) applies in view of the toroidal boundary conditions conserving linear momentum. However, for $d = 3$ and $\nu = 1$, Eq. (8) predicts that the state integral diverges for large N and that the collapse cannot reach a stationary state. The appearance of a stationary cluster is therefore a consequence of a regularization of the pair potential for small $|\mathbf{x}_i - \mathbf{x}_j|$ in ref. 51. For $1/r$-potential systems with short-range cutoff, the transition from a homogeneous gaseous state to the stationary clustered state is a phase transition of first order[52] with the critical point determined by the ultraviolet cutoff.[53] The situation is analogous to an attractive particle system interacting with a negative Gaussian potential.[54]

If one wants to apply statistical mechanics to gravitating particles, they have to be put in a box to prevent evaporation. But even then the volume of the energy shell will become infinite because it is infinitely extended in kinetic energy. In physical terms this means that, if the trajectory visits all regions on the energy shell, high kinetic energies will dominate and the systems will keep heating up. This is exactly what the computer simulations find, and also astrophysical facts point in this direction. Putting particle systems in a box means that one keeps the particle density high. Indeed, in the centers of galaxies, where such conditions prevail, one finds double stars with breath-taking velocities.

The application of statistical mechanics to gravitational systems has been criticized by van Kampen with the argument that the usual formalism does not apply due to the fact that the volume of the energy shell becomes infinite and, thus, the partition function diverges. This means that the tendency towards a negative specific heat becomes even more dramatic. Ordinarily, a negative specific heat implies that a system heats up when brought into contact with a heat bath. In the

case studied here, not even a heat bath is required for the systems, they do it just by themselves. However, for small-enough particle numbers this process may come to an end if additional constants of the motion exist, which effectively remove the singularity of the partition function.

The singular behavior of the partition function is in contradiction to nonrelativistic quantum mechanics, where the energy is bounded from below and where there is no difficulty in defining the thermodynamic functions for all N. Nonrelativistically, the possibility of a complete collapse does not exist in quantum mechanics.[55] Nevertheless, a trace of this instability survives in the existence of a negative specific heat in a certain range of energies. The essential physics of this fact is captured in a very simple model by Hertel and Thirring.[56] What is left of the vastness of the energy shell for high momenta is that its volume increases more than exponentially with energy, which implies a negative specific heat. Quasi-relativistically, for sufficiently high N the system collapses.[57,58] For a recent discussion we refer to the work of Kiessling.[59]

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